Chapter 6

Graphing Linear and Quadratic Functions

In This Chapter

- Enlisting basic graphing techniques
- Identifying and graphing the equation of a line
- Graphing quadratic functions using vertices and intercepts

Graphing equations is an important part of understanding just what a function or other relationship represents. The modern, handheld graphing calculators take care of many of the details, but you still need to have a general idea of what the graph should look like so you know how to select a viewing window and so you know if you've entered something incorrectly.

Identifying Some Graphing Techniques

You do most graphing in algebra on the *Cartesian coordinate system* — a grid-like system where you plot points depending on the position and signs of numbers. Within the Cartesian coordinate system (which is named for the philosopher and mathematician Rene Descartes), you can plug-and-plot points to draw a curve, or you can take advantage of knowing a little something about what the graphs should look like. In either case, the coordinates and points fit together to give you a picture.

Graphing curves can take as long as you like or be as quick as you like. If you take advantage of the characteristics of the curves you're graphing, you can cut down on the time it takes to graph and improve your accuracy. Two features that you can quickly recognize and solve for are the intercepts and symmetry of the graphs.

Finding x- and y-intercepts

The *intercepts* of a graph appear at the points where the graph crosses the axes. The graph of a curve may never cross an axis, but when it does, knowing the points that represent the intercepts is very helpful.

The x-intercepts always have the format $(h, 0)$ — the y-coordinate is 0 because the point is on the x-axis. The y-intercepts have the form $(0, k)$ — the x-coordinate is 0 because the point is on the y-axis. You find the x- and y-intercepts by letting y and x, respectively, equal 0. To find the x-intercept(s) of a curve, you set y equal to 0 and solve a given equation for x. To find the y-intercept(s) of a curve, you set x equal to 0 and solve the equation for y.

Find the intercepts of the graph of $y = -x^2 + x + 6$.

To find the x-intercepts, let $y = 0$; you then have the quadratic equation $0 = -x^2 + x + 6 = -(x^2 - x - 6)$. Solve this equation by factoring it into $0 = -(x - 3)(x + 2)$. You find two solutions, $x = 3$ and -2, so the two x-intercepts are $(3, 0)$ and $(-2, 0)$. (For more on factoring, see Chapters 1 and 3.)

To find the y-intercept, let $x = 0$. This gives you the equation $y = -0 + 0 + 6 = 6$. The y-intercept, therefore, is $(0, 6)$.

Reflecting on a graph's symmetry

A graph that's *symmetric* with respect to one of the axes appears to be a mirror image of itself on either side of the axis. A graph symmetric about the origin appears to be the same after a 180-degree turn. Figure 6-1 shows three curves and three symmetries: symmetry with respect to the y-axis (a), symmetry with respect to the x-axis (b), and symmetry with respect to the origin (c).

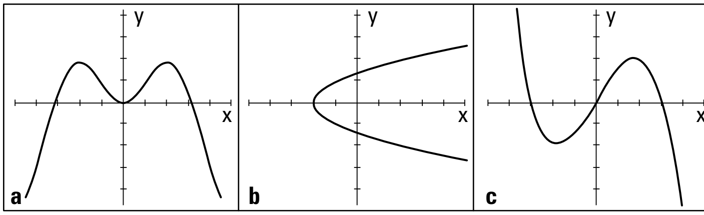

Figure 6-1: Symmetry in a graph makes for a pretty picture.

Recognizing that the graph of a curve has symmetry helps you sketch the graph and determine its characteristics. The following sections outline ways to determine, from a graph's equation, if symmetry exists.

With respect to the *y*-axis (even functions):

- If replacing every x with $-x$ doesn't change the value of y, the curve is the mirror image of itself over the y-axis. The graph contains the points (x, y) and $(-x, y)$.
- For example, the graph of the equation $y = x^4 - 3x^2 + 1$ is symmetric with respect to the y-axis. If you replace each x with $-x$, the equation remains unchanged. Replacing each x with $-x$, $y = (-x)^4 - 3(-x)^2 + 1 = x^4 - 3x^2 + 1$.

With respect to the *x*-axis:

- If replacing every y with $-y$ doesn't change the value of x, the curve is the mirror image of itself over the x-axis. The graph contains the points (x, y) and $(x, -y)$.
- For example, the graph of $x = \frac{10}{y^2 + 1}$ is symmetric with respect to the x-axis. When you replace each y with $-y$, the x-value remains unchanged.

With respect to the origin (odd functions):

- If replacing every variable with its opposite is the same as multiplying the entire equation by -1, the curve can rotate by 180 degrees about the origin and be its own image. The graph contains the points (x, y) and $(-x, -y)$.
- For example, the graph of $y = x^5 - 10x^3 + 9x$ is symmetric with respect to the origin. When you replace every x and y with $-x$ and $-y$, you get $-y = -x^5 + 10x^3 - 9x$, which is the same as multiplying everything through by -1.

Mastering the Graphs of Lines

Lines are some of the simplest graphs to sketch. It takes only two points to determine the one and only line that passes through them and goes on forever and ever in a space, so one simple method for graphing lines is to find two points — any two points — on the line. Another useful method is to use a point and the slope of the line. The method you choose is often just a matter of personal preference.

The slope of a line also plays a big role in comparing it with other lines that run parallel or perpendicular to it. The slopes are closely related to one another.

Determining the slope of a line

The *slope* of a line, designated by the letter m, has a complicated math definition, but it's basically a number — positive, negative, or zero; large or small — that tells you something about the steepness and direction of the line. The numerical value of the slope tells you if the line slowly rises or drops from left to right or dramatically soars or falls from left to right.

Characterizing a line's slope

A line can have a positive slope, a negative slope, a zero slope, or no slope at all. The greater the *absolute value* (the value of the number without regard to the sign; in other words, the distance of the number from 0) of a line's slope, the steeper the line is. For example, if the slope is a number between –1 and 1, the line is rather flat. A slope of 0 means that the line is absolutely horizontal.

A vertical line doesn't have a slope. This is tied to the fact that numbers go infinitely high, and math doesn't have a highest number — you just say *infinity.* Only an infinitely high number can represent a vertical line's slope, but usually, if you're talking about a vertical line, you just say that the slope doesn't exist.

Computing a line's slope

You can determine the slope of a line, m, if you know two points on the line.

You find the slope of the line that goes through the points (x_1, y_1) and (x_2, y_2) with the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Find the slope of the line through (–3, 2) and (4, –12).

Use the formula to get $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-12-2}{4-(-3)} = \frac{-14}{7} = -2$. This line is fairly steep — the absolute value of –2 is 2 — and it falls as it moves from left to right, which makes its slope negative.

When you use the slope formula, it doesn't matter which point you choose to be (x_1, y_1) — the order of the points doesn't matter — but you can't mix up the order of the two different coordinates. You can run a quick check by seeing if the coordinates of each point are above and below one another. Also, be sure that the *y*-coordinates are in the numerator; a common error is to have the difference of the y-coordinates in the denominator.

Describing two line equations

I offer two different forms for the equation of a line. The first is the *standard form,* written $Ax + By = C$, with the two variable terms on one side and the constant on the other side. The other form is the *slope-intercept form,* written $y = mx + b$; the *y*-value is set equal to the product of the slope, *m*, and *x* added to the *y*-intercept, *b*.

Standing up with the standard form

The standard form has more information about the line than may be immediately apparent. You can determine, just by looking at the numbers in the equation, the intercepts and slope of the line.

The line $Ax + By = C$ has

- An *x*-intercept of $\left(\frac{C}{A}, 0\right)$
- A *y*-intercept of $\left(0, \frac{C}{B}\right)$
- A slope of $m = -\frac{A}{B}$

Graph the line $4x + 3y = 12$ using the intercepts.

Plot the intercepts, $\left(\frac{C}{A},0\right)=\left(\frac{12}{4},0\right)=(3,0)$ and $\left(0,\frac{C}{B}\right)=\left(0,\frac{12}{3}\right)=(0,4)$.

Then draw the line through them. Figure 6-2 shows the two intercepts and the graph of the line. Note that the line falls as it moves from left to right, confirming the negative value of the slope from the formula $m=-\frac{A}{B}=-\frac{4}{3}$.

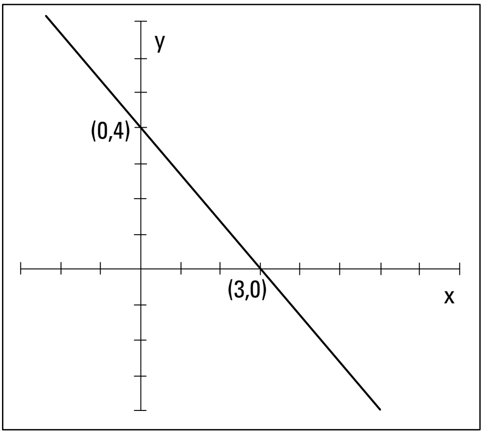

Figure 6-2: Graphing $4x+3y=12$, a line written in standard form, using its intercepts.

Sliding down the slope-intercept form

When the equation of a line is written in the slope-intercept form, $y = mx + b$, you have good information right at your fingertips. The coefficient of the x term, m, is the slope of the line. And the constant, b, is the y-value of the y-intercept. With these two bits of information, you can quickly sketch the line.

If you want to graph the line $y = 2x + 5$, for example, you first plot the y-intercept, (0, 5), and then "count off" the slope from that point moving to the right and then up or down. The slope of the line $y = 2x + 5$ is 2; think of the 2 as the slope fraction, with the y-coordinates on top and the x-coordinates on bottom. The slope then becomes $\frac{2}{1}$. So you move one unit to the right and then two units up, because the slope is positive.

Changing from one form to the other

You can graph lines by using the standard form or the slope-intercept form of the equations. If you prefer one form to the other — or if you need a particular form for an application you're working on — you can change the equations to your preferred form by performing simple algebra:

- To change the standard form to the slope-intercept form, you just solve for y.
- To change the slope-intercept form to the standard form, you rewrite the equation with the x and y terms on one side and then multiply through by a constant to create integer coefficients and a constant on the other side.

Identifying parallel and perpendicular lines

Lines are *parallel* when they never touch — no matter how far out you draw them. Lines are *perpendicular* when they intersect at a 90-degree angle. Both of these instances are fairly easy to spot when you see the lines graphed, but how can you be sure that the lines are truly parallel or that the angle is really 90 degrees and not 89.9 degrees? The answer lies in the slopes.

Consider two lines, $y = m_1x + b_1$ and $y = m_2x + b_2$.

REMEMBER

Two lines are *parallel* when their slopes are equal ($m_1 = m_2$). Two lines are *perpendicular* when their slopes are negative reciprocals of one another: $\left(m_2 = -\frac{1}{m_1}\right)$.

For example, the lines $y = 3x + 7$ and $y = 3x - 2$ are parallel. Both lines have a slope of 3, but their y-intercepts are different — one crosses the y-axis at 7 and the other at –2. The lines $y = -\frac{3}{8}x + 4$ and $y = \frac{8}{3}x - 2$ are perpendicular. The slopes are negative reciprocals of one another.

Coming to Terms with the Standard Form of a Quadratic

A *parabola* is the graph of a quadratic function. The graph is a nice, gentle, U-shaped curve that has points located an equal distance on either side of a line running up through its middle — called its *axis of symmetry.* Parabolas can be turned upward, downward, left, or right, but parabolas that represent functions only turn up or down. Here's the standard form for the quadratic function:

$$f(x) = ax^2 + bx + c$$

The coefficients (multipliers of the variables) a, b, and c are real numbers; a can't be equal to 0 because you'd no longer have a quadratic function. There's meaning in everything — or nothing!

Starting with "a" in the standard form

As the lead coefficient of the standard form of the quadratic function $f(x) = ax^2 + bx + c$, a provides important information:

- If a is positive, the graph of the parabola opens upward.
- If a is negative, the graph of the parabola opens downward.
- If a has an absolute value greater than 1, the graph of the parabola is "steep."
- If a has an absolute value less than 1, the graph of the parabola flattens.

Figure 6-3 shows some representatives of the different directions and forms that parabolas can take.

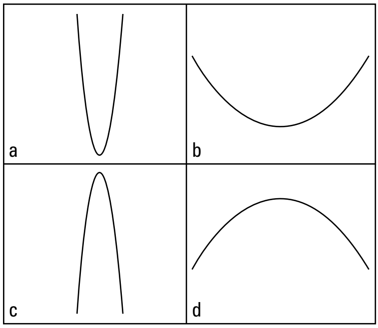

Figure 6-3: Parabolas opening up and down, appearing steep and flat.

The following equations of parabolas demonstrate for you the effect of the coefficient on the squared term:

- $\mathbf{y = 4x^2 - 3x + 2}$**:** You say that this parabola is steep and opens upward because the lead coefficient is positive and greater than 1.
- $\mathbf{y = -\frac{1}{3}x^2 + x - 11}$**:** You say that this parabola is flattened out and opens downward because the lead coefficient is negative, and the absolute value of the fraction is less than 1.
- $\mathbf{y = 0.002x^2 + 3}$**:** You say that this parabola is flattened out and opens upward because the lead coefficient is positive, and the decimal value is less than 1. In fact, the coefficient is so small that the flattened parabola almost looks like a horizontal line.

Following "a" with "b" and "c"

Much like the lead coefficient in the quadratic function (see the previous section), the terms *b* and *c* give you plenty of information. Mainly, the terms tell you a lot if they're *not* there.

- If the second coefficient, b, is 0, the parabola straddles the y-axis. The parabola's *vertex* — the highest or lowest point on the curve, depending on which way it faces — is on that axis, and the parabola is symmetric about the axis. The equation then takes the form $y = ax^2 + c$.
- If the last coefficient, c, is 0, the graph of the parabola goes through the origin — in other words, one of its intercepts is the origin. The equation then becomes $y = ax^2 + bx$, which you can easily factor into $y = x(ax + b)$.

Eyeing a Quadratic's Intercepts

The *intercepts* of a quadratic function (or any function) are the points where the graph of the function crosses the x- or y-axis.

Intercepts are very helpful when you're graphing a parabola. The points are easy to find because one of the coordinates is always 0. If you have the intercepts and the vertex, and you use the symmetry of the parabola, you have a good idea of what the graph looks like.

Finding the one and only y-intercept

The y-intercept of a quadratic function is $(0, c)$. A parabola with the standard equation $y = ax^2 + bx + c$ is a function, so by definition, only one y-value can exist for every x-value. When $x = 0$, then $y = c$ and the y-intercept is $(0, c)$.

To find the y-intercepts of the following functions, you let $x = 0$:

- $\mathbf{y = 4x^2 - 3x + 2}$**:** When $x = 0$, $y = 2$ (or $c = 2$). The y-intercept is (0, 2).
- $\mathbf{y = -x^2 - 5}$**:** When $x = 0$, $y = -5$ (or $c = -5$). Don't let the missing x term throw you. The y-intercept is (0, –5).
- $\mathbf{y = x^2 + 9x}$**:** When $x = 0$, $y = 0$. The equation provides no constant term; you could also say the missing constant term is 0. The y-intercept is (0, 0).

Getting at the x-intercepts

You find the x-intercepts of quadratics when you solve for the *zeros*, or solutions, of a quadratic equation and find real number answers. Parabolas with an equation of the standard form $y = ax^2 + bx + c$ open upward or downward and may or may not have x-intercepts; when the equation $0 = ax^2 + bx + c$ has no real solutions, then the graph has no x-intercepts.

The coordinates of all x-intercepts have zeros in them. An x-intercept's y-value is 0, and you write it in the form $(h, 0)$. How do you find the value of h? You let $y = 0$ in the general equation and then solve for x. You have two options to solve the equation $0 = ax^2 + bx + c$:

- Use the quadratic formula (see Chapter 3).
- Factor the expression and use the multiplication property of zero (MPZ; see Chapter 1).

Find the x-intercepts of $y = 3x^2 + 7x - 40$.

Set y equal to 0 and solve the quadratic equation by factoring:

$$0 = 3x^2 + 7x - 40 = (3x - 8)(x + 5)$$

So $x = \frac{8}{3}$ or $x = -5$.

The two x-intercepts are $\left(\frac{8}{3}, 0\right)$ and $(-5, 0)$.

This next example shows how you determine that an equation has no x-intercept.

Find the x-intercepts of $y = -2x^2 + 4x - 7$.

Set y equal to 0 and you find that the quadratic doesn't factor. Then you apply the quadratic formula.

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(-2)(-7)}}{2(-2)} \\ &= \frac{-4 \pm \sqrt{16 - (56)}}{-4} \\ &= \frac{-4 \pm \sqrt{-40}}{-4} \end{aligned}$$

You see that the value under the radical is negative; there are no real solutions. Alas, you find no *x*-intercept for this parabola.

Finding the Vertex of a Parabola

Quadratic functions, or parabolas, that have the standard form $y = ax^2 + bx + c$ are gentle, U-shaped curves that open either upward or downward. When the lead coefficient, a, is a positive number, the parabola opens upward, creating a *minimum value* for the function — the function values never go lower than that minimum. When a is negative, the parabola opens downward, creating a *maximum value* for the function — the function values never go higher than that maximum. The two extreme values, the minimum and maximum, occur at the parabola's *vertex.* The *y*-coordinate of the vertex gives you the numerical value of the extreme — its highest or lowest point. And the *x*-coordinate is part of the equation of the axis of symmetry.

Computing vertex coordinates

Finding the vertex of the parabola representing a quadratic function is as easy as *a*, *b*, *c* — without the *c*. Just insert the coefficients *a* and *b* into a formula.

The parabola $y = ax^2 + bx + c$ has its vertex when the *x*-value is equal to $\frac{-b}{2a}$. You plug in the *a* and *b* values from the equation to come up with the *x*-coordinate, and then you find the *y*-coordinate of the vertex by plugging this *x*-value into the equation and solving for *y*.

Find the coordinates of the vertex of $y = -3x^2 + 12x - 7$.

Solving for *x*, use the coefficients *a* and *b*:

$$x = \frac{-12}{2(-3)} = \frac{-12}{-6} = 2$$

You solve for *y* by putting the *x*-value back into the equation:

$$y = -3(2)^2 + 12(2) - 7 = -12 + 24 - 7 = 5$$

The coordinates of the vertex are (2, 5). This is the highest point for the parabola, because *a* is a negative number, which means the parabola opens downward from this point.

Linking up with the axis of symmetry

The *axis of symmetry* of a quadratic function is a vertical line that runs through the vertex of the parabola and acts as a mirror — half the parabola rests on one side of the axis, and half rests on the other. The x-value in the coordinates of the vertex appears in the equation for the axis of symmetry. For example, if a vertex has the coordinates (2, 3), the axis of symmetry is $x = 2$. All vertical lines have an equation of the form $x = h$. In the case of the axis of symmetry, the h is always the x-coordinate of the vertex.

Sketching a Graph from the Available Information

You have all sorts of information available when it comes to a quadratic function and its graph. You can use the intercepts, the opening, the steepness, the vertex, the axis of symmetry, or just some random points to plot the parabola. You don't really need all the pieces for each graph; as you practice sketching these curves, it becomes easier to figure out which pieces you need for different situations. The example I give, though, will use all the different possibilities — and each will just verify all the others.

EXAMPLE

Sketch the graph of $y = x^2 - 4x - 5$.

First, notice that the equation represents a parabola that opens upward, because the lead coefficient, *a*, is positive (+1). The y-intercept is (0, –5), which you get by plugging in 0 for x. If you set y equal to 0 to solve for the x-intercepts, you get $0 = x^2 - 4x - 5$, which factors into $0 = (x + 1)(x - 5)$. The x-intercepts are (–1, 0) and (5, 0).

The vertex is found using the formula for the x-coordinate of the vertex to get $x = \frac{-(-4)}{2(1)} = 2$. Plug the 2 into the formula for the parabola, and you find that the vertex is at (2, –9).

Use the axis of symmetry, which is $x = 2$, to find some points on either side — to help you with the shape. If you let $x = 6$, for example, you find that $y = 7$. This point is four units to the right of $x = 2$; four units to the left of $x = 2$ is $x = -2$. The corresponding point is found by putting –2 into the equation for the parabola; you get (–2, 7).

Use all that information in a graph to produce a sketch of the parabola (see Figure 6-4).

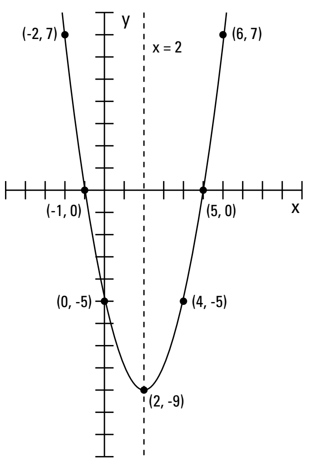

Figure 6-4: Using the various pieces of a quadratic as steps for sketching a graph ($y = x^2 - 4x - 5$).